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Schrödinger functional formalism with Ginsparg-Wilson fermion

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Abstract: The Schrödinger functional formalism is given as a field theory in a finite volume with a Dirichlet boundary condition in temporal direction. When one tries to construct this formalism with the Ginsparg-Wilson fermion including the overlap Dirac operator and the domain-wall fermion one easily runs into difficulties. The reason is that if the Dirichlet boundary condition is simply imposed on the Wilson Dirac operator D_W inside of the overlap Dirac operator an exponentially small eigenvalue appears in D_W, which affects the locality properties of the operator. In this paper we propose a new procedure to impose the Schrödinger functional Dirichlet boundary condition on the overlap Dirac operator using an orbifolding projection.

Keywords: Lattice Gauge Field Theories, Lattice QCD.

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1. Introduction

The Schrödinger functional (SF) is defined as a transition amplitude between two boundary states with finite time separation [1, 2]

$$Z = \left\langle C'; x_0 = T | C; x_0 = 0 \right\rangle = \int \mathcal{D}\Phi e^{-S[\Phi]} \tag{1.1}$$

and is written in a path integral representation of the field theory with some specific boundary condition.[1] One of applications of the SF is to define a renormalization scheme beyond perturbation theory, where the renormalization scale is given by a finite volume $T \times L^3 \sim L^4$ of the system. The formulation is already accomplished for the non-linear σ-model [5], the non-abelian gauge theory [6] and the QCD with the Wilson fermion [7, 8] including $\mathcal{O}(a)$ improvement procedure [9, 10]. (See ref. [11] for review.)

In this formalism several renormalization quantities like running gauge coupling [12 − 17], Z-factors and $\mathcal{O}(a)$ improvement factors [18 − 22] are extracted conveniently by using a Dirichlet boundary conditions for spatial component of the gauge field

$$A_k(x)|_{x_0=0} = C_k(\vec{x}), \qquad A_k(x)|_{x_0=T} = C'_k(\vec{x}) \tag{1.2}$$

[1]The same kind of transition amplitude was introduced in [3, 4] in order to implement the temporal gauge.

and for the quark fields

$$P_+\psi(x)|_{x_0=0} = \rho(\vec{x})\,, \qquad P_-\psi(x)|_{x_0=T} = \rho'(\vec{x})\,, \tag{1.3}$$
$$\overline{\psi}(x)P_-|_{x_0=0} = \overline{\rho}(\vec{x})\,, \qquad \overline{\psi}(x)P_+|_{x_0=T} = \overline{\rho}'(\vec{x})\,, \tag{1.4}$$
$$P_\pm = \frac{1\pm\gamma_0}{2}\,. \tag{1.5}$$

One of privilege of this Dirichlet boundary condition is that the system acquire a mass gap and there is no infra-red divergence problem.

We notice that the boundary condition is not free to set since it generally breaks symmetry of the theory and may affect renormalizability. However the field theory with Dirichlet boundary condition is shown to be renormalizable for the pure gauge theory [6]. And it is also the case for the Wilson fermion [8] by including a shift in the boundary fields.

Although it is essential to adopt Dirichlet boundary condition for a mass gap and renormalizability, it has a potential problem of zero mode in fermion system. For instance starting from a free lagrangian

$$\mathcal{L} = \overline{\psi}\left(\gamma_\mu\partial_\mu + m\right)\psi \tag{1.6}$$

with positive mass $m > 0$ and the Dirichlet boundary condition

$$P_-\psi|_{x_0=0} = 0\,, \qquad P_+\psi|_{x_0=T} = 0 \tag{1.7}$$

the zero eigenvalue equation $(\gamma_0\partial_0 + m)\,\psi = 0$ in temporal direction allows a solution

$$\psi = P_+e^{-mx_0} + P_-e^{-m(T-x_0)} \tag{1.8}$$

in $T \to \infty$ limit and a similar solution remains even for finite T with an exponentially small eigenvalue $\propto e^{-mT}$. In the SF formalism this solution is forbidden by adopting an "opposite" Dirichlet boundary condition (1.3) and the system has a finite gap even for $m = 0$ [7].

In the SF formalism of the Wilson fermion [7] we cut the Wilson Dirac operator at the boundary and the Dirichlet boundary condition is automatically chosen among

$$P_\pm\psi|_{x_0=0} = 0\,, \qquad P_\mp\psi|_{x_0=T} = 0 \tag{1.9}$$

depending on signature of the Wilson term. For example if we adopt a typical signature of the Wilson term

$$D_W = \gamma_\mu\frac{1}{2}\left(\nabla^*_\mu + \nabla_\mu\right) - \frac{a}{2}\nabla^*_\mu\nabla_\mu + M \tag{1.10}$$

the allowed Dirichlet boundary condition is the same as (1.3). In this case the zero mode solution can be forbidden by choosing a proper signature for the mass term; the mass should be kept positive $M \geq 0$ to eliminate the zero mode [7].

However this zero mode problem may become fatal in the Ginsparg-Wilson fermion including the overlap Dirac operator [23, 24] and the domain-wall fermion [25–28]. The overlap Dirac operator is defined by using the Wilson Dirac operator (1.10) as

$$D = \frac{1}{\overline{a}}\left(1 + D_W\frac{1}{\sqrt{D_W^\dagger D_W}}\right), \qquad \overline{a} = \frac{a}{|M|}\,. \tag{1.11}$$

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Here we notice that the Wilson fermion mass M should be kept negative in a range $-2 < M < 0$ to impose heavy masses on the doubler modes and a single massless mode to survive. As explained in the above when the Dirichlet boundary condition is imposed directly to the kernel D_W an exponentially small eigenvalue is allowed for this choice of the Wilson parameter and the mass. Nearly zero eigenvalue in D_W may break the locality of the overlap Dirac operator [29]. The situation is also quite similar in the domain-wall fermion. The same zero mode solution appears in the transfer matrix in fifth direction, which suppresses the dumping solution in fifth dimension and allow a chiral symmetry breaking term to appear in the Ward-Takahashi identity [27].

Since a naive formulation of the SF formalism by setting the Dirichlet boundary condition for the kernel D_W does not work, we need different procedure to impose boundary condition on the overlap Dirac fermion. In this paper we propose an orbifolding projection for this purpose. In section 2 we introduce an orbifolding in a continuum theory and show that the Dirac operator of the orbifolded theory satisfy the same SF boundary condition. Although this procedure may be applicable to the general Ginsparg-Wilson fermion we concentrate on the overlap Dirac fermion in section 3. Section 4 is devoted for conclusion.

2. Orbifolding for continuum theory

We notice a fact that the chiral symmetry is broken explicitly by the Dirichlet boundary condition (1.3) in the SF formalism. This should be also true in the overlap Dirac operator; the Ginsparg-Wilson relation should be broken in some sense, which was not accomplished in a naive formulation. We would adopt this property as a criterion of the SF formalism.

Then we remind a fact that an orbifolded field theory is equivalent to a field theory with some specific boundary condition. Since it is possible to break chiral symmetry by an orbifolding projection in general, it may be able to represent the SF formalism as an orbifolded theory. In this section we search for an orbifolding projection which is not consistent with chiral symmetry and provide the same SF Dirichlet boundary condition (1.3) and (1.4) at fixed points.

We consider a massless free fermion on $S^1 \times \mathbb{R}^3$

$$\mathcal{L} = \overline{\psi}(x)\gamma_\mu\partial_\mu\psi(x)\,, \tag{2.1}$$

where the anti-periodic boundary condition is set in temporal direction of length $2T$

$$\psi(\vec{x}, x_0 + 2T) = -\psi(\vec{x}, x_0)\,, \qquad \overline{\psi}(\vec{x}, x_0 + 2T) = -\overline{\psi}(\vec{x}, x_0)\,. \tag{2.2}$$

The orbifolding S^1/Z_2 in temporal direction is accomplished by identifying the negative time with the positive one $x_0 \leftrightarrow -x_0$. Identification of the fermion field is given by using a symmetry transformation including the time reflection

$$\psi(x) \to \Sigma\psi(x)\,, \qquad \overline{\psi}(x) \to \overline{\psi}(x)\Sigma\,, \qquad \Sigma = i\gamma_5\gamma_0 R\,, \tag{2.3}$$

where R is a time reflection operator

$$R\psi(\vec{x}, x_0) = \psi(\vec{x}, -x_0)\,. \tag{2.4}$$

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R has two fixed points $x_0 = 0, T$, where $x_0 = 0$ is a symmetric and $x_0 = T$ is an anti-symmetric fixed point because of the anti-periodicity

$$R\psi(\vec{x}, 0) = \psi(\vec{x}, 0)\,, \qquad R\psi(\vec{x}, T) = -\psi(\vec{x}, T)\,. \tag{2.5}$$

It is free to add any internal symmetry transformation for the identification and we use the chiral symmetry of the massless fermion

$$\psi(x) \to -i\gamma_5\psi(x)\,, \qquad \overline{\psi}(x) \to -\overline{\psi}(x)i\gamma_5\,. \tag{2.6}$$

Combining (2.3) and (2.6) we have the orbifolding symmetry transformation

$$\psi(x) \to -\Gamma\psi(x)\,, \qquad \overline{\psi}(x) \to \overline{\psi}(x)\Gamma\,, \qquad \Gamma = \gamma_0 R\,. \tag{2.7}$$

The orbifolding of the fermion field is given by selecting the following symmetric subspace

$$\Pi_+\psi(x) = 0\,, \qquad \overline{\psi}(x)\Pi_- = 0\,, \qquad \Pi_\pm = \frac{1 \pm \Gamma}{2}\,. \tag{2.8}$$

We notice that this orbifolding projection provides the proper homogeneous SF Dirichlet boundary condition at fixed points $x_0 = 0, T$

$$P_+\psi(x)|_{x_0=0} = 0\,, \qquad P_-\psi(x)|_{x_0=T} = 0\,, \tag{2.9}$$
$$\overline{\psi}(x)P_-|_{x_0=0} = 0\,, \qquad \overline{\psi}(x)P_+|_{x_0=T} = 0\,. \tag{2.10}$$

The orbifolded action is given by the same projection

$$S = \frac{1}{2}\int d^4x\overline{\psi}(x)D_{\mathrm{SF}}\psi(x)\,, \qquad D_{\mathrm{SF}} = \Pi_+\not{\partial}\Pi_-\,, \tag{2.11}$$

where factor 1/2 is included since the temporal direction is doubled compared to the original SF formalism. We notice that the chiral symmetry is broken explicitly for D_{SF} by the projection.

Now we have two comments. Since the Schrödinger functional of the pure gauge theory is already well established [6] we treat the gauge field as an external field and adopt a configuration which is time reflection invariant

$$A_0(\vec{x}, -x_0) = -A_0(\vec{x}, x_0)\,, \qquad A_i(\vec{x}, -x_0) = A_i(\vec{x}, x_0) \tag{2.12}$$

and satisfy the SF boundary condition (1.2) simultaneously. We set periodic boundary condition for the gauge field

$$A_\mu(\vec{x}, x_0 + 2T) = A_\mu(\vec{x}, x_0)\,. \tag{2.13}$$

Second comment is on the mass term. Although the mass term is not consistent with the chiral symmetry, we can find a symmetric mass term under the orbifolding transformation. A requirement is that the mass matrix M should anti-commute with orbifolding operator $\{M, \Gamma\} = 0$. One of the candidate is a time dependent mass $M = m\eta(x_0)$ with anti-

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symmetric and periodic step function

$$\eta(-x_0) = -\eta(x_0)\,, \qquad \eta(x_0+2T) = \eta(x_0)\,,$$
$$\eta(x_0) = 1 \quad \text{for} \quad 0 < x_0 < T\,. \tag{2.14}$$

Now the Dirac operator becomes

$$D(m) = \gamma_\mu \left(\partial_\mu - iA_\mu(x)\right) + m\eta(x_0)\,, \tag{2.15}$$

which has the orbifolding symmetry

$$\{D(m), \Gamma\} = 0\,. \tag{2.16}$$

The orbifolded action is defined in the same manner as the free theory

$$S = \frac{1}{2}\int d^4x \overline{\psi}(x) D_{\mathrm{SF}}(m)\psi(x)\,, \qquad D_{\mathrm{SF}}(m) = \Pi_+ D(m)\Pi_-\,. \tag{2.17}$$

We can show that this system with gauge interaction is equivalent to the original QCD with SF boundary condition [7]. We consider the bulk region of the temporal direction: $-T < x_0 < 0$ and $0 < x_0 < T$. In this region the orbifolding condition (2.8) becomes

$$\psi(x_0) = -\gamma_0\psi(-x_0)\,, \qquad \overline{\psi}(x_0) = \overline{\psi}(-x_0)\gamma_0 \tag{2.18}$$

to identify the fields in positive and negative time. Combining with the anti-commutative property of the Dirac operator (2.16) we can show that contribution to the action from the bulk region is written as

$$\begin{aligned} S_{\mathrm{bulk}} &= \frac{1}{2}\int_{\mathrm{bulk}} d^4x \overline{\psi}(x)\Pi_+ D(m)\Pi_-\psi(x) \\ &= \int_0^T dx_0 d^3\vec{x}\overline{\psi}(x)\left[\gamma_\mu\left(\partial_\mu - iA_\mu(x)\right) + m\right]\psi(x)\,, \end{aligned} \tag{2.19}$$

which is nothing but the QCD action introduced in ref. [7] except for the surface term, which will be discussed later. Since the bulk action and the boundary condition for the orbifold construction is exactly the same as those of QCD with SF boundary condition, the spectrum and the propagator should be uniquely determined to be equivalent to those of ref. [7] and ref. [10]. One can easily check this fact at tree level.

3. Orbifolding for overlap Dirac fermion

Application of the orbifolding procedure is straightforward to the Ginsparg-Wilson fermions including the overlap Dirac operator [23, 24], the domain-wall fermion [25−28] and the perfect action [30−32] which possess both the time reflection symmetry

$$[\Sigma, D] = 0 \tag{3.1}$$

and the lattice chiral symmetry [33] stemming from the Ginsparg-Wilson relation [34]

$$\gamma_5 D + D\gamma_5 = \overline{a} D\gamma_5 D\,. \tag{3.2}$$

In this subsection we concentrate on the overlap Dirac operator (1.11), for which the time reflection symmetry (3.1) comes from that of the Wilson Dirac operator $[\Sigma, D_W] = 0$.

3.1 Orbifolding construction of Dirichlet boundary

As in the continuum case we consider a massless fermion on a lattice $2N_T \times N_L^3$ with anti-periodic boundary condition in temporal direction (2.2). We use an orbifolding S^1/Z_2 in temporal direction. Identification of the fermion field is given by using the time reflection (2.3) and the chiral symmetry of the overlap Dirac fermion [33]

$$\psi(x) \to -i\widehat{\gamma}_5\psi(x)\,, \qquad \overline{\psi}(x) \to -\overline{\psi}(x)i\gamma_5\,, \qquad \widehat{\gamma}_5 = \gamma_5\left(1-\overline{a}D\right), \tag{3.3}$$

where the gauge field is treated as an external field and we adopt a time reflection symmetric configuration

$$U_k(\vec{x},x_0) = U_k(\vec{x},-x_0)\,, \qquad U_0(\vec{x},x_0) = U_0^\dagger(\vec{x},-x_0-1)\,, \tag{3.4}$$

satisfying the SF Dirichlet boundary condition simultaneously

$$U_k(\vec{x},0) = W_k(\vec{x})\,, \qquad U_k(\vec{x},N_T) = W_k'(\vec{x})\,. \tag{3.5}$$

Combining (2.3) and (3.3) we have the orbifolding symmetry transformation

$$\psi(x) \to -\widehat{\Gamma}\psi(x)\,, \qquad \overline{\psi}(x) \to \overline{\psi}(x)\Gamma\,, \qquad \widehat{\Gamma} = \Gamma(1-\overline{a}D)\,, \tag{3.6}$$

where Γ is the same as the continuum one (2.7). We notice that starting from the time reflection symmetry of the Dirac operator (3.1) and the Ginsparg-Wilson relation (3.2) we have another GW relation for Γ

$$\Gamma D + D\Gamma = \overline{a}D\Gamma D \tag{3.7}$$

and Γ hermiticity

$$\Gamma D\Gamma = D^\dagger\,. \tag{3.8}$$

The operator $\widehat{\Gamma}$ has a property $\widehat{\Gamma}^2 = 1$ like Γ and can be used to define a projection operator in the following.

The orbifolding identification of the fermion field is given in the same way with slightly different projection operator

$$\widehat{\Pi}_+\psi(x) = 0\,, \qquad \overline{\psi}(x)\Pi_- = 0\,, \qquad \widehat{\Pi}_\pm = \frac{1\pm\widehat{\Gamma}}{2}\,, \tag{3.9}$$

which turn out to be the SF Dirichlet boundary condition (2.9) and (2.10) at fixed points in the continuum limit. Using the time reflection symmetry (3.1) we can easily show that the projection operators Γ and $\widehat{\Gamma}$ do not have an "index"[2]

$$\mathrm{tr}\Gamma = \mathrm{tr}\hat{\Gamma} = 0 \tag{3.10}$$

and furthermore we can find a local unitary transformation

$$u = \frac{1+\Sigma}{2}(1-\overline{a}D) + \frac{1-\Sigma}{2}, \quad u' = \gamma_5 u\gamma_5\,, \tag{3.11}$$

[2]The same property is satisfied for the chiral index $\mathrm{tr}\widehat{\gamma}_5 = \mathrm{tr}(\widehat{\gamma}_5\Sigma^2) = -\mathrm{tr}\widehat{\gamma}_5 = 0$.

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which connects $\widehat{\Gamma}$ and Γ as

$$\widehat{\Gamma} = u^\dagger \Gamma u\,, \qquad \widehat{\Gamma} = u' \Gamma u'^\dagger\,. \tag{3.12}$$

The projection operator $\widehat{\Pi}_\pm$ spans essentially the same Hilbert sub-space as $\Pi_\pm$. We notice that this unitary operator connects $\widehat{\gamma}_5$ and γ_5 in a similar way

$$\widehat{\gamma}_5 = u^\dagger \gamma_5 u\,, \qquad \widehat{\gamma}_5 = u' \gamma_5 u'^\dagger\,. \tag{3.13}$$

The physical quark operator is defined to transform in a same manner as the continuum under chiral rotation,

$$\delta q(x) = \gamma_5 q(x)\,, \qquad \delta \overline{q}(x) = \overline{q}(x)\gamma_5\,. \tag{3.14}$$

Since we have a unitary operator u and u' we have several ways to define a physical quark field from GW fermion fields ψ and $\overline{\psi}$.[3] For example

$$q(x) = \left(1 - \frac{\overline{a}}{2} D\right)\psi(x), \quad \overline{q}(x) = \overline{\psi}(x)\,, \tag{3.15}$$

$$q(x) = u\psi(x)\,, \qquad \overline{q}(x) = \overline{\psi}(x)\,, \tag{3.16}$$

$$q(x) = u'^\dagger\psi(x)\,, \qquad \overline{q}(x) = \overline{\psi}(x)\,. \tag{3.17}$$

These three definitions are not independent but connected with $u + u'^\dagger = (2 - \overline{a}D)$. The orbifolding of the physical quark field becomes the same as that of the continuum theory

$$\Pi_+ q(x) = 0\,, \qquad \overline{q}(x)\Pi_- = 0\,. \tag{3.18}$$

The massless orbifolded action is given by

$$S = \frac{1}{2} a^4 \sum \overline{\psi} D_{\rm SF} \psi, \quad D_{\rm SF} = \Pi_+ D \widehat{\Pi}_-\,. \tag{3.19}$$

We have four comments here. (i) It should be emphasized that the SF Dirac operator $D_{\rm SF}$ is local since it is constructed by multiplying local objects only. (ii) The massless SF Dirac operator $D_{\rm SF}$ does not satisfy the chiral Ginsparg-Wilson relation (3.2) and the chiral symmetry is broken by projection as was expected. (iii) Although two different projection operators Γ and $\widehat{\Gamma}$ are used from the left and right of $D_{\rm SF}$ this does not bring the problem we encountered in the chiral gauge theory since these two operators are connected by the unitary transformation u or u'. (iv) Since we have some ambiguity in defining the mass term we concentrate on the massless case and postpone the massive theory to the later sub-section.

It is clear that (3.19) is a well defined lattice regularization of the SF QCD with gauge interaction since the Dirac operator is local and the lattice action (3.19) converges to the continuum one (2.17) in $a \to 0$ limit including the boundary condition. We shall see explicit scaling behavior for free theory in the following two sub-sections.

[3] One may suppose that a local Dirac operator can be constructed with exact chiral symmetry by adopting the physical quark operator of (3.16) and (3.17), where the effective action of q, $\overline{q}$ is kept local. However the unitary transformation u contains the time reflection operator R and furthermore the gauge configuration is fixed to be time reflection invariant. There is no translation invariance in the physical quark effective theory and this is consistent with the Nielsen-Ninomiya no-go theorem.

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3.2 Eigenvalue of free Dirac operator

We consider an eigenvalue problem of the massless SF Dirac operator

$$D_{\mathrm{SF}} = \Pi_+ D \widehat{\Pi}_- \tag{3.20}$$

at tree level. This Dirac operator connects two different Hilbert sub-space as in the continuum [7]

$$D_{\mathrm{SF}} : \widehat{\mathcal{H}}_+ \to \mathcal{H}_-, \quad D_{\mathrm{SF}}^\dagger : \mathcal{H}_- \to \widehat{\mathcal{H}}_+ , \tag{3.21}$$

$$\widehat{\mathcal{H}}_+ = \left\{ \psi | \widehat{\Pi}_+ \psi = 0 \right\}, \qquad \mathcal{H}_- = \{ \psi | \Pi_- \psi = 0 \} . \tag{3.22}$$

We need to introduce the "doubled" hermitean Dirac operator

$$\mathcal{D} = \begin{pmatrix} & D_{\mathrm{SF}} \\ D_{\mathrm{SF}}^\dagger & \end{pmatrix}, \tag{3.23}$$

which connects the same Hilbert space $\mathcal{H}_- \oplus \widehat{\mathcal{H}}_+ \to \mathcal{H}_- \oplus \widehat{\mathcal{H}}_+$. As in the continuum the eigenvalue problem is solved on a two component vector

$$\Psi = \begin{pmatrix} \psi_- \\ \widehat{\psi}_+ \end{pmatrix}, \qquad \psi_- \in \mathcal{H}_-, \quad \widehat{\psi}_+ \in \widehat{\mathcal{H}}_+ \tag{3.24}$$

and the eigenvalue equation is written in a following form

$$D_{\mathrm{SF}} \widehat{\psi}_+ = \lambda \psi_- , \qquad D_{\mathrm{SF}}^\dagger \psi_- = \lambda \widehat{\psi}_+ \tag{3.25}$$

with a real eigenvalue λ.

A candidate of the eigen-function in each Hilbert sub-space $\widehat{\mathcal{H}}_+$ and $\mathcal{H}_-$ is given by

$$\widehat{\psi}_+(x) = u^\dagger \psi_+(x) , \qquad \psi_+(x) = f_+(x_0) v e^{i\vec{p}\vec{x}} , \qquad \psi_-(x) = f_-(x_0) w e^{i\vec{p}\vec{x}} , \tag{3.26}$$

where $f_\pm$ are given to satisfy $\Gamma f_\pm = \mp f_\pm$ as

$$f_+(x_0) = \alpha \left(P_+ \sin p_0 x_0 + P_- \cos p_0 x_0 \right) , \tag{3.27}$$

$$f_-(x_0) = \alpha \left(P_+ \cos p_0 x_0 + P_- \sin p_0 x_0 \right) \tag{3.28}$$

with

$$p_0 = \frac{2n-1}{2N_T} \pi , \qquad n = \{ -N_T + 1, \cdots, N_T \} \tag{3.29}$$

for anti-periodicity in $2N_T$. v and w are four component vectors in spinor space. In the following we consider the operator

$$D_{\mathrm{SF}} u^\dagger = \Pi_+ \left\{ \frac{1}{2} \left(D - D^\dagger \right) - \Sigma \frac{1}{2} \left(D + D^\dagger \right) \right\} \Pi_- \tag{3.30}$$

and its operation on ψ_+.

We start from the fact that ψ_+ is an eigen-function of the operator $D_W^\dagger D_W$ and $D_W + D_W^\dagger$ with eigenvalues λ_W^2 and $2W$ where

$$\lambda_W^2 = \sum_\mu \sin^2 a p_\mu + W^2 , \qquad W = -|M| + \sum_\mu \left(1 - \cos a p_\mu \right) . \tag{3.31}$$

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On the other hand $D_W - D_W^\dagger$ operates as

$$\frac{1}{2}\left(D_W - D_W^\dagger\right)\psi_+(x) = (\sin ap_0 f_-(x_0) + i\gamma_i \sin ap_i f_+(x_0))\, ve^{i\vec{p}\vec{x}}. \tag{3.32}$$

Combining these two and taking similar manipulation for $uD_{\rm SF}^\dagger$ on ψ_- the eigenvalue equation is re-written as

$$\begin{aligned} D_{\rm SF}\widehat{\psi}_+(x) &= f_-(x_0)\,(A_0 + i\gamma_i A_i + i\gamma_5 B)\, ve^{i\vec{p}\vec{x}} = \lambda\psi_-(x)\,, & (3.33)\\ D_{\rm SF}^\dagger\psi_-(x) &= u^\dagger f_+(x_0)\,(A_0 - i\gamma_i A_i - i\gamma_5 B)\, we^{i\vec{p}\vec{x}} = \lambda\widehat{\psi}_+(x) & (3.34)\end{aligned}$$

with

$$A_\mu = \frac{\sin ap_\mu}{\overline{a}\sqrt{\lambda_W^2}}, \quad B = \frac{1}{\overline{a}}\left(1 + \frac{W}{\sqrt{\lambda_W^2}}\right). \tag{3.35}$$

Here we used $\Sigma = i\gamma_5\Gamma$ and a property $\Gamma f_\pm = \mp f_\pm$. At last we solve the eigenvalue equation in the spinor space and get the eigenvalue

$$\lambda^2 = A_\mu^2 + B^2\,. \tag{3.36}$$

This eigenvalue agrees with that of the massless continuum theory [7] in $a \to 0$ limit combined with the discretization condition (3.29).

3.3 Free propagator

The fermion propagator is formally given by the inverse of the Dirac operator

$$G_{\rm SF}(x,y) = 2\left(D_{\rm SF}^{-1}\right)_{x,y} = 2\left(\widehat{\Pi}_-\frac{1}{D}\Pi_+\right)_{x,y}, \tag{3.37}$$

where inverse is defined in the Hilbert sub-space $\widehat{\mathcal{H}}_+$ or $\mathcal{H}_-$ as

$$D_{\rm SF}D_{\rm SF}^{-1} = \Pi_+\,, \qquad D_{\rm SF}^{-1}D_{\rm SF} = \widehat{\Pi}_-\,. \tag{3.38}$$

At tree level this propagator can be written in a simple form as

$$\begin{aligned} G_{\rm SF} &= 2D^\dagger\Pi_+\frac{1}{DD^\dagger}\Pi_+ = D^\dagger\left(P_+G_L + P_-G_R\right), & (3.39)\\ G_{\rm R/L}(x,y) &= \frac{1}{N_L^3}\sum_{\vec{p}} e^{i\vec{p}(\vec{x}-\vec{y})}G_{\rm R/L}(x_0,y_0;\vec{p})\,, & (3.40)\\ G_{\rm R}(x_0,y_0;\vec{p}) &= \frac{1}{2aN_T}\sum_{n=-N_T+1}^{N_T}\frac{1}{A_\mu^2+B^2}\left(e^{ip_0(x_0-y_0)} - e^{ip_0(x_0+y_0)}\right), & (3.41)\\ G_{\rm L}(x_0,y_0;\vec{p}) &= \frac{1}{2aN_T}\sum_{n=-N_T+1}^{N_T}\frac{1}{A_\mu^2+B^2}\left(e^{ip_0(x_0-y_0)} + e^{ip_0(x_0+y_0)}\right), & (3.42)\end{aligned}$$

where p_0 satisfy the quantization condition (3.29).

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If we adopt the definition (3.15) for the physical quark field its propagator is given by

$$\langle q(x)\overline{q}(y)\rangle = \left[\left(1-\frac{\overline{a}}{2}D\right)G_{\rm SF}\right](x,y)\,. \tag{3.43}$$

The proper Dirichlet boundary condition [10] is trivially satisfied for this quark propagator

$$P_+\,\langle q(x)\overline{q}(y)\rangle\,|_{x_0=0} = 0\,, \qquad P_-\,\langle q(x)\overline{q}(y)\rangle\,|_{x_0=T} = 0\,, \tag{3.44}$$

$$\langle q(x)\overline{q}(y)\rangle\,|_{y_0=0}P_- = 0\,, \qquad \langle q(x)\overline{q}(y)\rangle\,|_{y_0=T}P_+ = 0 \tag{3.45}$$

because of the projection $\Pi_\pm$. At tree level the propagator takes the form

$$\begin{aligned} a^3\sum_{\vec{x}} e^{-i\vec{p}(\vec{x}-\vec{y})}\,\langle q(x)\overline{q}(y)\rangle &= \frac{1}{2aN_T}\sum_{n=-N_T+1}^{N_T}\left(\frac{-i\gamma_\mu A_\mu+B}{A_\mu^2+B^2}-\frac{\overline{a}}{2}\right)\times \\ &\quad\times e^{ip_0x_0}\left\{\left(e^{-ip_0y_0}+e^{ip_0y_0}\right)P_+ + \left(e^{-ip_0y_0}-e^{ip_0y_0}\right)P_-\right\}, \end{aligned} \tag{3.46}$$

which can be shown to approach to the continuum SF propagator [10] without any $\mathcal{O}(a)$ term.

3.4 Phase of Dirac determinant

In general the determinant of the SF Dirac operator is not real for the overlap fermion since there is no γ_5 Hermiticity. Instead we have a following "Hermiticity" relation

$$\gamma_5 u D_{\rm SF} u^\dagger \gamma_5 = D_{\rm SF}^\dagger\,, \qquad \gamma_5 u'^\dagger D_{\rm SF} u' \gamma_5 = D_{\rm SF}^\dagger\,. \tag{3.47}$$

However one cannot conclude reality from this relation since the SF Dirac operator connects different Hilbert sub-space as in (3.21) and the determinant cannot be evaluated directly with $D_{\rm SF}$. We need to make a "hermitean" Dirac operator which connects the same Hilbert sub-space in order to define the Dirac determinant. This is accomplished by $u^\dagger\gamma_5$ or $u'\gamma_5$, which turns out to be γ_5 in the continuum. We define

$$H_{\rm SF} = D_{\rm SF}u^\dagger\gamma_5 = \Pi_+ D u^\dagger \gamma_5 \Pi_+ \ : \qquad \mathcal{H}_- \to \mathcal{H}_-\,, \tag{3.48}$$

$$H'_{\rm SF} = D_{\rm SF}u'\gamma_5 = \Pi_+ D u' \gamma_5 \Pi_+ \ : \qquad \mathcal{H}_- \to \mathcal{H}_-\,. \tag{3.49}$$

The determinant is evaluated on the sub-space $\mathcal{H}_-$

$$\det_{\{\mathcal{H}_-\}} H_{\rm SF} = \det\left(\Pi_+ D u^\dagger \gamma_5 \Pi_+ + \Pi_-\right), \tag{3.50}$$

$$\det_{\{\mathcal{H}_-\}} H'_{\rm SF} = \det\left(\Pi_+ D u' \gamma_5 \Pi_+ + \Pi_-\right), \tag{3.51}$$

where the right hand side is understood to be evaluated in the full Hilbert space by filling the opposite sub-space $\mathcal{H}_+$ with unity.

The phase of the determinant is given as follows

$$\left(\det_{\{\mathcal{H}_-\}} H_{\rm SF}^{(\prime)}\right)^* = e^{-2i\phi^{(\prime)}}\left(\det_{\{\mathcal{H}_-\}} H_{\rm SF}^{(\prime)}\right), \tag{3.52}$$

$$e^{-2i\phi} = \det_{\{\mathcal{H}_-\}}\left(\gamma_5 u\right)^2 = \det u, \tag{3.53}$$

$$e^{-2i\phi'} = \det_{\{\mathcal{H}_-\}}\left(\gamma_5 u'^\dagger\right)^2 = \det u^\dagger = e^{2i\phi}, \tag{3.54}$$

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which is not real in general. In the second equality of (3.53) and (3.54) we used a relation

$$\Sigma = \omega \Gamma \omega^\dagger\,, \qquad \Gamma = \omega^\dagger \Sigma \omega\,, \qquad \omega = e^{i\frac{\pi}{4}\gamma_5}\,. \tag{3.55}$$

The determinant of the unitary operator u is given by a product of eigenvalues λ_n of the overlap Dirac operator

$$\det u = \prod_{n\in\{+\}} (1 - a\lambda_n)\,, \tag{3.56}$$

where product is taken over a sub-space in which the eigenvalue of $\Sigma = +1$ and the conjugate eigenvalue λ_n^* does not necessarily belongs to this sub-space.

However we notice that this complexity of the Dirac determinant is not an essential problem since the phase is an $\mathcal{O}(a)$ irrelevant effect and disappears in the continuum limit. Furthermore if we consider variation of the phase

$$\delta_{\epsilon(x)}\phi = \frac{i}{2}\mathrm{tr}\delta_{\epsilon(x)} u u^{-1} = -\frac{i}{4} a\mathrm{tr}\left[\Sigma \delta_{\epsilon(x)} D \left(1 - aD^\dagger\right)\right] \tag{3.57}$$

under a local variation of the link variable

$$\delta_{\epsilon(x)} U_\mu(x) = a\epsilon_\mu(x) U_\mu(x) \tag{3.58}$$

we can show that $\delta_{\epsilon(x)}\phi$ is localized at the boundary. Since Σ contains time reflection R and both of the operator δD and $\left(1 - aD^\dagger\right)$ are local, the trace in (3.57) has a contribution only at the boundary. Contribution from the bulk is suppressed exponentially by the locality property.

For practical application to numerical simulation this phase problem should be settled. This is possible for even number of flavours by using two definitions of "hermitean" Dirac operator (3.48) (3.49) and a fact that the phase is opposite for these definitions as shown in (3.53) (3.54). The phase can be absorbed into re-definition of the fermion fields. Explicit form of the two flavour hermitean Dirac operator is

$$H_{\mathrm{SF}}^{(2)} = \begin{pmatrix} D_{\mathrm{SF}} & \\ & D_{\mathrm{SF}} \end{pmatrix} U^\dagger \gamma_5 \tau^{1,2}\ : \qquad \mathcal{H}_- \oplus \mathcal{H}_- \to \mathcal{H}_- \oplus \mathcal{H}_-\,, \tag{3.59}$$

where the unitary matrix is defined as

$$U = \begin{pmatrix} u & \\ & u'^\dagger \end{pmatrix} \tag{3.60}$$

to act on flavour space and τ^a is the Pauli matrix. γ_5 is also understood as two by two on flavour space. The Dirac operator is exactly Hermite for this definition: $H_{\mathrm{SF}}^{(2)\dagger} = H_{\mathrm{SF}}^{(2)}$ and the determinant is real.

For single flavour case it is not still clear how to solve this practical problem. However the determinant is real at tree level and one may expect that the phase is settled as one approaches to the continuum limit.

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3.5 Surface term

When extracting the renormalization factors of fermions it is convenient to consider a operator involving the boundary source fields

$$\zeta(\vec{x}) = \frac{\delta}{\delta\overline{\rho}(\vec{x})}, \qquad \overline{\zeta}(\vec{x}) = -\frac{\delta}{\delta\rho(\vec{x})}, \tag{3.61}$$

$$\zeta'(\vec{x}) = \frac{\delta}{\delta\overline{\rho}'(\vec{x})}, \qquad \overline{\zeta}'(\vec{x}) = -\frac{\delta}{\delta\rho'(\vec{x})}, \tag{3.62}$$

where $\rho, \cdots, \overline{\rho}'$ are boundary values of the fermion fields given in (1.3) and (1.4). Coupling of the boundary value to the bulk dynamical fields was naturally introduced in the Wilson fermion [7]. However this is not the case for our construction since the boundary value vanishes with the orbifolding projection.

In this paper we regard the boundary value as an external source field and introduce its coupling with the bulk fields according to the criteria: the coupling terms (surface terms) are local and reproduce the same form of the correlation function between the boundary fields in the continuum limit. Here we define the boundary values on the physical quark fields

$$P_+q(x)|_{x_0=0} = \rho(\vec{x}), \qquad P_-q(x)|_{x_0=N_T} = \rho'(\vec{x}), \tag{3.63}$$

$$\overline{q}(x)P_-|_{x_0=0} = \overline{\rho}(\vec{x}), \qquad \overline{q}(x)P_+|_{x_0=N_T} = \overline{\rho}'(\vec{x}). \tag{3.64}$$

One of candidates of the surface term is

$$S_{\text{surface}} = a^3\sum_{\vec{x}}\Big(- \overline{\rho}(\vec{x})P_-q(x)|_{x_0=0} - \overline{q}(x)P_+\rho(\vec{x})|_{x_0=0} - \\ - \overline{\rho}'(\vec{x})P_+q(x)\big|_{x_0=N_T} - \overline{q}(x)P_-\rho'(\vec{x})\big|_{x_0=N_T}\Big), \tag{3.65}$$

where q and $\overline{q}$ are active dynamical fields on the boundary.

According to ref. [10] we introduce the generating functional

$$Z_F\left[\overline{\rho}', \rho'; \overline{\rho}, \rho; \overline{\eta}, \eta; U\right] = \int D\psi D\overline{\psi}\exp\Big\{-S_F\left[U, \overline{\psi}, \psi; \overline{\rho}', \rho', \overline{\rho}, \rho\right] \\ + a^4\sum_x\left(\overline{\psi}(x)\eta(x) + \overline{\eta}(x)\psi(x)\right)\Big\}, \tag{3.66}$$

where $\eta(x)$ and $\overline{\eta}(x)$ are source fields for the fermion fields and the total action S_F is given as a sum of the bulk action (3.19) and the surface term (3.65). We notice that the fermion fields ψ and $\overline{\psi}$ obey the orbifolding condition (3.9). We decompose the fermion fields into classical and quantum components

$$\psi(x) = \psi_{\text{cl}}(x) + \chi(x), \qquad \overline{\psi}(x) = \overline{\psi}_{\text{cl}}(x) + \overline{\chi}(x) \tag{3.67}$$

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and insert it into the generating functional. The correlation functions between the boundary fields are derived with the same procedure as ref. [10] by making use of this decomposition.

$$\left\langle\psi(x)\overline{\psi}(y)\right\rangle = G_{\mathrm{SF}}(x,y)\,, \tag{3.68}$$

$$\langle q(x)\overline{q}(y)\rangle = \left[\left(1-\frac{\overline{a}}{2}D\right)G_{\mathrm{SF}}\right](x,y)\,, \tag{3.69}$$

$$\left\langle q(x)\overline{\zeta}(\vec{y})\right\rangle = \left[\left(1-\frac{\overline{a}}{2}D\right)G_{\mathrm{SF}}\right](x,y)P_+\Big|_{y_0=0}\,, \tag{3.70}$$

$$\left\langle q(x)\overline{\zeta}'(\vec{y})\right\rangle = \left[\left(1-\frac{\overline{a}}{2}D\right)G_{\mathrm{SF}}\right](x,y)P_-\Big|_{y_0=N_T}\,, \tag{3.71}$$

$$\langle\zeta(\vec{x})\overline{q}(y)\rangle = P_-\left[\left(1-\frac{\overline{a}}{2}D\right)G_{\mathrm{SF}}\right](x,y)\Big|_{x_0=0}\,, \tag{3.72}$$

$$\left\langle\zeta'(\vec{x})\overline{q}(y)\right\rangle = P_+\left[\left(1-\frac{\overline{a}}{2}D\right)G_{\mathrm{SF}}\right](x,y)\Big|_{x_0=N_T}\,, \tag{3.73}$$

$$\left\langle\zeta(\vec{x})\overline{\zeta}(\vec{y})\right\rangle = P_-\left[\left(1-\frac{\overline{a}}{2}D\right)G_{\mathrm{SF}}\right](x,y)P_+\Big|_{x_0=0,y_0=0}\,, \tag{3.74}$$

$$\left\langle\zeta(\vec{x})\overline{\zeta}'(\vec{y})\right\rangle = P_-\left[\left(1-\frac{\overline{a}}{2}D\right)G_{\mathrm{SF}}\right](x,y)P_-\Big|_{x_0=0,y_0=N_T}\,, \tag{3.75}$$

$$\left\langle\zeta'(\vec{x})\overline{\zeta}(\vec{y})\right\rangle = P_+\left[\left(1-\frac{\overline{a}}{2}D\right)G_{\mathrm{SF}}\right](x,y)P_+\Big|_{x_0=N_T,y_0=0}\,, \tag{3.76}$$

$$\left\langle\zeta'(\vec{x})\overline{\zeta}'(\vec{y})\right\rangle = P_+\left[\left(1-\frac{\overline{a}}{2}D\right)G_{\mathrm{SF}}\right](x,y)P_-\Big|_{x_0=N_T,y_0=N_T}\,. \tag{3.77}$$

Here we adopted the physical quark field of (3.15). The propagator G_{SF} is defined in (3.37). We notice that the above propagators between the boundary fields and physical quark fields approach to the continuum SF boundary propagator without any $\mathcal{O}(a)$ term.

3.6 Mass term

The mass term may be introduced with the same procedure as the continuum theory. We consider a mass matrix M which is consistent with the orbifolding symmetry

$$\Gamma M + M\widehat{\Gamma} = 0\,. \tag{3.78}$$

Since the orbifolding transformation is the same as the continuum one on the physical quark fields, a naive candidate is to couple the continuum mass matrix $m\eta(x_0)$ to the physical scalar density consisting of $q(x)$ and $\overline{q}(x)$. Corresponding to various definition of the quark fields (3.15)-(3.17) we have several definitions of the mass term

$$\mathcal{L}_m = m\overline{\psi}\eta\left(1-\frac{\overline{a}}{2}D\right)\psi\,,\qquad m\overline{\psi}\eta u\psi\,,\qquad m\overline{\psi}\eta u'^{\dagger}\psi\,, \tag{3.79}$$

where η is an anti-symmetric step function (2.14) on lattice.

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However we encounter a problem with this naive definition of mass term, since the massive Dirac operator does not satisfy the "Hermiticity" relation (3.47). The phase of the Dirac determinant becomes mass dependent although it is still irrelevant $\mathcal{O}(a)$ term.

In order to avoid this unpleasant situation we propose even flavors formulation. For two flavors case we define the two by tow Dirac operator as

$$D_{\rm SF}^{(2)}(m) = \begin{pmatrix} D_{\rm SF}(m)_1 & \\ & D_{\rm SF}(m)_2 \end{pmatrix}, \tag{3.80}$$

where

$$D_{\rm SF}(m)_1 = \Pi_+ \left(D + m\eta \left(1 - \frac{\overline{a}}{2} D \right) \right) \widehat{\Pi}_- \tag{3.81}$$

$$D_{\rm SF}(m)_2 = \Pi_+ \left(D + m \left(1 - \frac{\overline{a}}{2} D \right) u'\eta u'^{\dagger} \right) \widehat{\Pi}_- \,. \tag{3.82}$$

A "hermitean" relation can be found for this two flavors Dirac operator as

$$D_{\rm SF}^{(2)}(m)^{\dagger} = \tau^{1,2}\gamma_5 U D_{\rm SF}^{(2)}(m) U^{\dagger}\gamma_5\tau^{1,2}\,, \tag{3.83}$$

where U is defined in (3.60).

The hermitean Dirac operator can be defined to connect the same Hilbert sub-space as

$$H_{\rm SF}^{(2)}(m) = D_{\rm SF}^{(2)}(m) U^{\dagger}\gamma_5\tau^{1,2} \qquad : \qquad \mathcal{H}_- \oplus \mathcal{H}_- \to \mathcal{H}_- \oplus \mathcal{H}_-\,, \tag{3.84}$$

which is re-written in a trivially hermitean form by a unitary matrix V

$$H_{\rm SF}^{(2)}(m) = V \begin{pmatrix} & D_{\rm SF}(m)_1 \\ D_{\rm SF}(m)_1^{\dagger} & \end{pmatrix} V^{\dagger}\,, \qquad V = \begin{pmatrix} 1 & \\ & \gamma_5 u \end{pmatrix}. \tag{3.85}$$

The determinant of this Dirac operator is evaluated in a single Hilbert sub-space

$$\det_{\{\mathcal{H}_-\oplus\mathcal{H}_-\}} H_{\rm SF}^{(2)}(m)$$

and becomes real.

For the other mass term using the physical quark fields of (3.16) and (3.17) we also have two flavors Dirac operator with

$$D_{\rm SF}^{(2)}(m) = \begin{pmatrix} \Pi_+ \left(D + m\eta u \right) \widehat{\Pi}_- & \\ & \Pi_+ \left(D + m\eta u'^{\dagger} \right) \widehat{\Pi}_- \end{pmatrix}, \tag{3.86}$$

which satisfy the same "Hermiticity" relation (3.83).

Here we notice that U(2) flavour symmetry is broken to $\mathrm{U}(1)_V \times \mathrm{U}(1)_3$ by mass term like chirally twisted Wilson fermion. However the symmetry is recovered in massless limit and $m = 0$ simulation is possible for relatively small box size we expect that this flavour symmetry breaking is not a serious problem.

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3.7 γ_5 mass term

We have another candidate of the mass term which is consistent with the orbifolding symmetry (3.78). That is the γ_5 mass term. In order to be able to define the hermitean Dirac operator we need two flavors also in this case. The orbifolded action becomes

$$S = \frac{1}{2} a^4 \sum \overline{\psi} D_{\mathrm{SF}}(m_5) \psi \tag{3.87}$$

with the SF Dirac operator in the two flavors space

$$D_{\mathrm{SF}}(m_5) = \Pi_+ \left(D + i\gamma_5 \tau^3 m_5 \left(1 - \frac{\overline{a}}{2} D \right) \right) \widehat{\Pi}_- \,. \tag{3.88}$$

The hermitean conjugate is expressed as

$$D_{\mathrm{SF}}(m_5)^\dagger = \tau^{1,2}\gamma_5 u D_{\mathrm{SF}}(m_5) u^\dagger \gamma_5 \tau^{1,2} = \tau^{1,2}\gamma_5 u'^\dagger D_{\mathrm{SF}}(m_5) u' \gamma_5 \tau^{1,2} \tag{3.89}$$

and we can easily show that the phase of the determinant is mass independent in this case. Furthermore we can absorb the phase into the fields and define the hermitean Dirac operator in the same manner

$$H_{\mathrm{SF}}(m_5) = D_{\mathrm{SF}}(m_5) U^\dagger \gamma_5 \tau^{1,2} \,, \tag{3.90}$$

for which we have exact Hermiticity $H_{\mathrm{SF}}(m_5)^\dagger = H_{\mathrm{SF}}(m_5)$ and real determinant.

Up to now the hermitean Dirac operator (3.59) (3.84) (3.90) contains gauge dependent unitary matrix $U^\dagger$, which may be an obstacle for practical application. This problem may be solved by chirally twisting the fields and adopting different boundary condition in temporal direction. We consider following chiral rotation of the fields

$$\psi = e^{-i\frac{\pi}{4}\widehat{\gamma}_5\tau^3} \psi' \,, \qquad \overline{\psi} = \overline{\psi}' e^{-i\frac{\pi}{4}\gamma_5\tau^3} \,. \tag{3.91}$$

The action is re-written as

$$S = \frac{1}{2} a^4 \sum \overline{\psi}' \overline{D}_{\mathrm{SF}}(m) \psi' \,, \tag{3.92}$$

$$\overline{D}_{\mathrm{SF}}(m) = \widetilde{\Pi}_- D(m) \widetilde{\Pi}_- \,, \qquad D(m) = D + m \left(1 - \frac{\overline{a}}{2} D \right) , \tag{3.93}$$

where the projection operator $\widetilde{\Pi}_\pm$ is defined by using the time reversal operator Σ of (2.3) and third component of the Pauli matrix

$$\widetilde{\Pi}_\pm = \frac{1 \pm \Sigma\tau^3}{2} \,. \tag{3.94}$$

We should notice that the new fields ψ' and $\overline{\psi}'$ obey a different orbifolding condition

$$\widetilde{\Pi}_+ \psi'(x) = 0 \,, \qquad \overline{\psi}'(x) \widetilde{\Pi}_+ = 0 \tag{3.95}$$

and satisfy a different condition at the boundary

$$\widetilde{P}_+ \psi'(x) \Big|_{x_0=0} = 0 \,, \qquad \widetilde{P}_- \psi'(x) \Big|_{x_0=T} = 0 \,, \tag{3.96}$$

$$\overline{\psi}'(x) \widetilde{P}_+ \Big|_{x_0=0} = 0 \,, \qquad \overline{\psi}'(x) \widetilde{P}_- \Big|_{x_0=T} = 0 \tag{3.97}$$

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with a projection

$$\widetilde{P}_\pm = \frac{1 \pm \widetilde{\Gamma}}{2}, \qquad \widetilde{\Gamma} = i\gamma_5\gamma_0\tau^3 . \tag{3.98}$$

Since the boundary condition is completely different, this formalism may not offer the same SF renormalization scheme. However it may be possible to define a new renormalization scheme with this theory. This formulation gives an equivalent partition function at least for the massless case connected by the exact chiral symmetry. This system has a gap caused by the discretization of p_0 in (3.29) and it may be feasible to make use of the same good property of the SF formalism in defining the renormalization scale.

It is clear that this Dirac operator gives real determinant with hermitean relation

$$\overline{D}_{\rm SF}(m)^\dagger = \gamma_5\tau^{1,2}\overline{D}_{\rm SF}(m)\gamma_5\tau^{1,2} . \tag{3.99}$$

The flavour symmetry is broken to $U(1)_V \times U(1)_3$ as other massive theories and $SU(2)_f$ symmetry in massless limit is guaranteed in a form of "Ginsparg-Wilson relation"

$$\gamma_5\tau^{1,2}\overline{D}_{\rm SF}(0) + \overline{D}_{\rm SF}(0)\gamma_5\tau^{1,2} = a\overline{D}_{\rm SF}(0)\gamma_5\tau^{1,2}\overline{D}_{\rm SF}(0) . \tag{3.100}$$

We notice that the projector $\Sigma\tau^3$ commute with the Dirac operator

$$\left[D(m), \Sigma\tau^3\right] = 0 \tag{3.101}$$

and the eigenvalue of this new system becomes the same as that of the ordinary massive overlap Dirac operator $D(m)$ with half numbers of degeneracy. The propagator is defined as

$$\overline{G}_{\rm SF}(x,y) = 2\left(\overline{D}_{\rm SF}(m)^{-1}\right)_{x,y} = 2\left(\widetilde{\Pi}_- \frac{1}{D(m)}\widetilde{\Pi}_-\right)_{x,y} , \tag{3.102}$$

which takes a simple form at tree level

$$\begin{aligned} a^3\sum_{\vec{x}} e^{-i\vec{p}\vec{x}}\overline{G}_{\rm SF}(x,y) = {} & \frac{1}{2aN_T}\sum_{n=-N_T+1}^{N_T}\frac{-i\gamma_\mu\overline{A}_\mu + M}{\overline{A}_\mu^2 + M^2}\times \\ & \times e^{ip_0x_0}\left\{\left(e^{-ip_0y_0} + e^{ip_0y_0}\right)\widetilde{P}_- + \left(e^{-ip_0y_0} - e^{ip_0y_0}\right)\widetilde{P}_+\right\}, \end{aligned} \tag{3.103}$$

where

$$\overline{A}_\mu = \left(1 - \frac{\overline{a}}{2}m\right)A_\mu , \qquad M = m + \left(1 - \frac{\overline{a}}{2}m\right)B \tag{3.104}$$

and A_μ, B are defined in (3.35). The temporal momentum p_0 has a discretized value of (3.29).

4. Conclusion

In this paper we propose a new procedure to introduce the SF Dirichlet boundary condition for general fermion fields. In the former formulation the boundary condition is introduced by cutting the Dirac operator at the boundary [7]. For the Wilson fermion the boundary condition is automatically decided depending on the signature of the Wilson parameter r. However this formulation produces an exponentially small eigenvalue in the kernel D_W of

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the overlap Dirac operator because the relative signature of the Wilson parameter r and the mass parameter M is opposite. This may cause breaking of the locality of the overlap Dirac operator or breaking of the axial Ward-Takahashi identity in the domain-wall fermion.

Instead of cutting the Dirac operator we focus on a fact that the chiral symmetry is broken explicitly in the SF formalism by the boundary condition and adopt it as a criterion of the procedure. We also notice that an orbifolded field theory is equivalent to a field theory with some specific boundary condition. We search for the orbifolding symmetry which is not consistent with the chiral symmetry and reproduces the SF Dirichlet boundary condition on the fixed points. We found that the orbifolding S^1/Z_2 in temporal direction including the time reflection, the chiral rotation and the anti-periodicity serves this purpose well.

Application of this procedure to the overlap Dirac operator is straightforward since this system has both the time reflection and the chiral symmetry. The eigenvalue of the Dirac operator and the propagator are derived at tree level, which are shown to agree with the continuum results in $a \to 0$ limit.

We found a technical problem that the Dirac determinant is complex. However this is not essential since the phase of the determinant is an irrelevant $\mathcal{O}(a)$ term and its variation is localized at the boundary. Furthermore we can absorb this phase into re-definition of the fermion fields for even flavours case and explicit form of hermitean Dirac operator was given for two flavours. For single flavour case it is not still clear how to deal with this problem. However the determinant becomes real at tree level and one may expect that the phase is settled as one approaches to the continuum limit. Numerical simulation is needed to observe scaling behavior of the complex phase.

The mass term may still have a problem. If we define the mass term in an ordinary way to couple to the physical quark field with kink function $\eta(x_0)$ the phase of the Dirac determinant becomes mass dependent. In order to avoid this problem we need even number of flavours, for which the phase can be absorbed into the fermion field. The chirally twisted mass term serves for the same purpose. This mass term is consistent with the orbifolding symmetry and gives mass independent phase of the determinant, which can also be absorbed by re-definition of fields. The most satisfactory solution may be to introduce the chirally twisted boundary condition, for which the Dirac determinant is naturally real without any re-definition of fields. We should notice that the $U(2)$ flavor symmetry is broken to $U(1)_V \times U(1)_f$ for these two flavors formulation. However the symmetry is recovered in massless limit and $m = 0$ simulation is possible for relatively small box size the flavour symmetry breaking may not be a serious problem.

The SF formalism with the domain-wall fermion can be formulated in the same way [35] since the symmetry is exactly the same as the overlap Dirac operator [28]. Application of this procedure to the Wilson fermion is not trivial since the Wilson term does not anti-commute with Γ operator but commute with it. In order to be consistent with the orbifolding symmetry we may need to introduce the step function η of (2.14) in the Wilson term. However the Wilson term vanishes on the boundary $x_0 = 0, T$ and the doublers appear. We may need to vanish all the fields at the boundary, with which we reproduce the same SF Dirac operator as the original one [7]. Introduction of γ_5 into the Wilson term may be another way to apply this procedure.

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